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AN A-STABLE EXTENDED TRAPEZOIDAL RULE FOR THE INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

RIAZ A. USMANI
Department of Applied Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada

and

RAVI P. AGARWAL
Department of Mathematics, National University of Singapore, Kent Ridge 0511, Singapore

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Abstract—We examine a single-step implicit-integration algorithm which is obtained by a modification of the well-known trapezoidal rule. The obtained new method is a third-order numerical process and preserves the property of A-stability of the trapezoidal rule. Numerical examples involving stiff linear systems of first-order differential equations are also included to demonstrate the practical usefulness of this new integration procedure.

1. INTRODUCTION

The trapezoidal rule for the integration of the initial-value problem

$$\frac{dy}{dx} = f(x, y); \quad y(a) = A, \; a \le x \le b, \tag{1.1}$$

is given by

$$z_{n+1} = z_n + \frac{h}{2}[f(x_n, z_n) + f(x_{n+1}, z_{n+1})], \quad n = 0, 1, 2 \ldots, \tag{1.2}$$

where the points x_n are defined by

$$x_n = a + nh; \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

and z_n is believed to be a closed approximation to the number $y_n = y(x_n)$, where $y(x)$ is the unique solution of (1.1). The number z_{n+1} can be obtained directly by solving the nonlinear eqn (1.2) or one can predict z_{n+1} by another method, such as Euler's method, namely

$$z_{n+1}^{(0)} = z_n + hf(x_n, z_n) \tag{1.4}$$

and then the predicted value can be improved to any desired degree of accuracy by the scheme

$$z_{n+1}^{(j+1)} = z_n + \frac{h}{2}[f(x_n, z_n) + f(x_{n+1}, z_{n+1}^{(j)})], \quad j = 0, 1, 2, \ldots. \tag{1.5}$$

These well-known facts are extensively detailed in the literature (for example, see [1], p. 29; [2], p. 199; [3], p. 10).

Dahlquist (1963) defined a method to be A-stable if the numerical solution z_n of the differential equation $y' = \lambda y$ where Re $(\lambda) < 0$ approaches zero as $n \to \infty$. It is known that the trapezoidal rule (1.2) is A-stable ([1], p. 43). Furthermore, the convergence of z_n to zero as $n \to \infty$ is monotonic or oscillatory accordingly as $h\lambda \in (-2, 0)$ or $(-\infty, -2)$, where $\lambda < 0$.

A one-step method is said to be L-stable if it is A-stable and in addition, when applied to the scalar differential equation $y' = \lambda y$ where Re $(\lambda) < 0$ yields

$$z_{n+1} = R(h\lambda)z_n, \tag{1.6}$$

where $|R(h\lambda)| \to 0$ as Re $(h\lambda) \to -\infty$. It is easily verified that (1.2) is not L-stable ([3], p. 235).

The purpose of this paper is to study an extension of the trapezoidal rule. We derive our method in the next section.

2. EXTENDED TRAPEZOIDAL RULE

We define the extended trapezoidal rule for the solution of (1.1) in the form

$$z_{n+1} = z_n + h[\alpha_0 f(x_n, z_n) + \alpha_1 f(x_{n+1}, z_{n+1}) + \alpha_2 f(x_{n+2}, \hat{z}_{n+2})] \tag{2.1}$$

where

$$\hat{z}_{n+2} = \beta_0 z_n + \beta_1 z_{n+1} + h[\beta_2 f(x_n, z_n) + \beta_3 f(x_{n+1}, z_{n+1})], \quad n = 0, 1, 2, \ldots . \tag{2.2}$$

If we expand the right side of (2.2) in the Taylor series and equate the coefficients up to the terms $O(h^3)$, we obtain the following system of equations:

$$\begin{aligned} \beta_0 + \beta_1 &= 1, \\ -2\beta_0 - \beta_1 + \beta_2 + \beta_3 &= 0, \\ 2\beta_0 + \tfrac{1}{2}\beta_1 - 2\beta_2 - \beta_3 &= 0, \\ -\tfrac{4}{3}\beta_0 - \tfrac{1}{6}\beta_1 + 2\beta_2 + \tfrac{1}{2}\beta_3 &= 0. \end{aligned}$$

The unique solution of the above system is $\beta_0 = 5$, $\beta_1 = -4$, $\beta_2 = 2$, $\beta_3 = 4$ and hence

$$\hat{z}_{n+2} = \psi(z_n, z_{n+1}) = 5z_n - 4z_{n+1} + h[2f(x_n, z_n) + 4f(x_{n+1}, z_{n+1})] \tag{2.3}$$

has the truncation error T_1 which is of order 4. Indeed, an easy computation provides

$$T_1 = \tfrac{1}{6}h^4(20y^{(4)}(p_1) - y^{(4)}(p_2) - 16y^{(4)}(p_3) - 4y^{(4)}(p_4)) \tag{2.4}$$

where $x_n < p_i < x_{n+2}$, $i = 1, 2, 3, 4$.

Similarly, we expand the right side of (2.1) in the Taylor series and equate the coefficients up to the terms $O(h^3)$, to obtain

$$\begin{aligned} \alpha_0 + \alpha_1 + \alpha_2 &= 1, \\ \alpha_0 - \alpha_2 &= \tfrac{1}{2}, \\ \alpha_0 + \alpha_2 &= \tfrac{1}{3}. \end{aligned}$$

The above system provides $\alpha_0 = \frac{5}{12}$, $\alpha_1 = \frac{2}{3}$, $\alpha_2 = -\frac{1}{12}$ and hence

$$z_{n+1} = \phi(z_n, z_{n+1}, \hat{z}_{n+2}) = z_n + \frac{h}{12}[5f(x_n, z_n) + 8f(x_{n+1}, z_{n+1}) - f(x_{n+2}, \hat{z}_{n+2})] \tag{2.5}$$

has the truncation error T_2 which is also of order 4 and

$$T_2 = \tfrac{1}{72}h^4(3y^{(4)}(q_1) - 5y^{(4)}(q_2) - y^{(4)}(q_3)) - \tfrac{1}{12}hT_1\frac{\partial f}{\partial y}(x_{n+2}, z_{n+2} + \theta T_1), \tag{2.6}$$

where $x_n < q_i < x_{n+2}$, $i = 1, 2, 3$ and $0 < \theta < 1$.

The pair of formulas (2.3), (2.5) form the basis of our extended trapezoidal rule. From this pair z_{n+1} is obtained by following an iterative process, namely

$$\begin{aligned} \hat{z}_{n+2}^{(j)} &= \psi(z_n, z_{n+1}^{(j)}), \\ z_{n+1}^{(j+1)} &= \phi(z_n, z_{n+1}^{(j)}, \hat{z}_{n+2}^{(j)}), \quad j = 0, 1, 2, \ldots . \end{aligned} \tag{2.7}$$

For starting the iterative process (2.7) we need an initial guess $z_1^{(0)}$ for z_1 which can be obtained from the Taylor series

$$z_1^{(0)} = A + hf(a, A) + \frac{h^2}{2}[f_x(a, A) + f(a, A)f_y(a, A)] + \cdots . \tag{2.8}$$

The process (2.7) is repeated until no further change occurs in the value of z_1 or meets the desired degree of accuracy. The value $\hat{z}_2^{(j)}$ obtained by (2.7) in the course of evaluating z_1 gives an initial approximation $z_2^{(0)}$ for z_2. The process (2.7) is now used for $n = 1$ with this $z_2^{(0)}$ to compute z_2 and so on.

The main ideas linked with the extended trapezoidal rule have been studied earlier by Urabe[4] and Usmani[5] but in their work the corresponding formulas depend on $f(x, y)$ and $g(x, y) = y'' = f_x(x, y) + f(x, y)f_y(x, y)$. However, if the function $g(x, y)$ cannot be evaluated on a digital machine with fair amount of ease, then the corresponding numerical methods are of little practical importance. Since our formulas depend only on the function $f(x, y)$ the present extended trapezoidal rule (2.3), (2.5) seems to have much future.

3. CONVERGENCE FACTOR

Subtracting (2.3) from the first equation of (2.7), we find

$$\begin{aligned} \hat{z}_{n+2}^{(j)} - \hat{z}_{n+2} &= \psi(z_n, z_{n+1}^{(j)}) - \psi(z_n, z_{n+1}) \\ &= -4(z_{n+1}^{(j)} - z_{n+1}) + 4h[f(x_{n+1}, z_{n+1}^{(j)}) - f(x_{n+1}, z_{n+1})], \end{aligned} \tag{3.1}$$

and hence

$$|\hat{z}_{n+2}^{(j)} - \hat{z}_{n+2}| \leq 4(1 + hL)|z_{n+1}^{(j)} - z_{n+1}|$$

where $L = \sup\limits_{\substack{a \leq x \leq b \\ -\infty < y < \infty}} \left| \dfrac{\partial f}{\partial y}(x, y) \right|$.

Similarly, on subtracting (2.5) from the second equation of (2.7), we get

$$\begin{aligned} z_{n+1}^{(j+1)} - z_{n+1} &= \phi(z_n, z_{n+1}^{(j)}, \hat{z}_{n+2}^{(j)}) - \phi(z_n, z_{n+1}, \hat{z}_{n+2}) \\ &= \frac{h}{12}[8f(x_{n+1}, z_{n+1}^{(j)}) - f(x_{n+2}, \hat{z}_{n+2}^{(j)}) - 8f(x_{n+1}, z_{n+1}) + f(x_{n+2}, \hat{z}_{n+2})] \end{aligned}$$

and hence

$$|z_{n+1}^{(j+1)} - z_{n+1}| \leq \frac{hL}{12}[8|z_{n+1}^{(j)} - z_{n+1}| + |\hat{z}_{n+2}^{(j)} - \hat{z}_{n+2}|]. \tag{3.2}$$

Using (3.1) in (3.2), we obtain

$$|z_{n+1}^{(j+1)} - z_{n+1}| \leq C|z_{n+1}^{(j)} - z_{n+1}|, \quad j = 0, 1, 2, \ldots , \tag{3.3}$$

where

$$C = hL + \tfrac{1}{3}h^2L^2. \tag{3.4}$$

The constant C is usually referred to as convergence factor.

By elementary manipulations the inequality (3.3) provides that

$$\left|z_{n+1}^{(j+1)} - z_{n+1}\right| \leq C^{j+1}\left|z_{n+1}^{(0)} - z_{n+1}\right|.$$

Thus, the iterative process (2.7) is convergent if $C < 1$, or

$$0 < hL < \frac{(-3 + \sqrt{21})}{2} \simeq 0.791. \tag{3.5}$$

4. ANALYSIS OF THE DISCRETIZATION ERROR

From the exact equations

$$y_{n+2} = \psi(y_n, y_{n+1}) + T_1 \tag{4.1}$$

and

$$y_{n+1} = \phi(y_n, y_{n+1}, y_{n+2}) + T_2, \tag{4.2}$$

we subtract (2.3) and (2.5) respectively to obtain

$$\hat{e}_{n+2} = 5e_n - 4e_{n+1} + h[2g_n e_n + 4g_{n+1}e_{n+1}] + T_1 \tag{4.3}$$

and

$$e_{n+1} = e_n + \frac{h}{12}[5g_n e_n + 8g_{n+1}e_{n+1} - g_{n+2}\hat{e}_{n+2}] + T_2, \tag{4.4}$$

where $e_n = y_n - z_n$, $\hat{e}_n = \hat{y}_n - \hat{z}_n$ and from the mean value theorem we write $f(x_n, y_n) - f(x_n, z_n) = g_n e_n$ so that g_n is the value of $f_y(x, y)$ evaluated at a point intermediate to (x_n, y_n) and (x_n, z_n).

Now, eliminating $\hat{e}_{n+2}$ from (4.3) and (4.4) we obtain the error equation

$$e_{n+1} = \frac{\alpha}{\beta} e_n + \frac{1}{\beta} T_3, \tag{4.5}$$

where

$$\alpha = 1 + \frac{5h}{12} g_n - \frac{5h}{12} g_{n+2} - \frac{2h^2}{12} g_n g_{n+2},$$

$$\beta = 1 - \frac{8h}{12} g_{n+1} - \frac{4h}{12} g_{n+2} + \frac{4h^2}{12} g_{n+1} g_{n+2},$$

$$T_3 = T_2 - \frac{h}{12} g_{n+1} T_1.$$

From (4.5), it follows that

$$|e_{n+1}| \leq \frac{1 + \frac{5}{6}hL + \frac{1}{6}h^2L^2}{1 - C}|e_n| + \frac{h^4(9 + 82hL)M_4}{72(1 - C)}, \tag{4.6}$$

where

$$M_4 = \max_{a \leq x \leq b}\left|\frac{d^4y}{dx^4}\right|.$$

Table 1.

x_n	z_n	MAE
1.0	1.1111110	0.45×10^{-8}
2.0	1.2499999	0.14×10^{-7}
3.0	1.4285713	0.33×10^{-7}
4.0	1.6666665	0.76×10^{-7}
5.0	1.999999	0.19×10^{-6}
6.0	2.499999	0.51×10^{-6}
7.0	3.33333	0.17×10^{-5}
8.0	4.99999	0.94×10^{-5}

Now, using standard inductive arguments in inequality (4.6), we obtain

$$|e_n| \leq \frac{h^4(9 + 82hL)M_4}{72(1 - C)} \frac{e^{n\alpha} - 1}{\alpha} \tag{4.7}$$

where

$$\alpha = \frac{h(11 + 3hL)L}{6(1 - C)}.$$

Inequality (4.7) is the same as

$$|e_n| \leq \frac{h^3(9 + 82hL)M_4}{12(11 + 3hL)L} \left[\exp\left(\frac{(b - a)(11 + 3hL)L}{6(1 - C)} \right) - 1 \right], \tag{4.8}$$

and hence $|e_n| = O(h^3)$. These considerations are stated in the following result.

THEOREM 1

The method defined by (2.3) and (2.5) for the numerical integration of the initial-value problem (1.1) is convergent and of order 3.

Example 1. Consider the initial-value problem

$$\frac{dy}{dx} = \tfrac{1}{10}y^2; \quad y(0) = 1, \; 0 \leq x \leq 8,$$

for which $y(x) = 10/(10 - x)$ is the exact solution. We solve this initial-value problem with $h = 2^{-m}$, $m = 2(1)6$. The numerical results are summarized for $h = \frac{1}{32}$ in Table 1. We observe that maximum absolute error (MAE) for this initial-value problem with $h = \frac{1}{32}$ is 0.93747×10^{-5}. Furthermore, the MAE for the same problem over the same range of integration with $h = \frac{1}{64}$ is 0.11579×10^{-5}. Thus, we find that on halving the step size h from $\frac{1}{32}$ to $\frac{1}{64}$ the MAE is approximately reduced by a factor $\frac{1}{8}$, which verifies the inequality (4.8).

In Table 1 all the entries in the second column are correct to the number of digits recorded after the decimal point.

5. ABSOLUTE STABILITY

In order to examine the present method for the absolute stability, we consider the differential equation $y' = \lambda y$. For this equation, from (2.3) and (2.5) it follows that

$$z_{n-2} = 5z_n - 4z_{n-1} + h\lambda(2z_n + 4z_{n+1}) \tag{5.1}$$

Table 2. Effect of absolute stability

λ	h	z_n	error $= e^{\lambda} - z_n$
-10	1.0	-0.353383	0.353428
	0.1	0.000033	0.116×10^{-4}
	0.01	0.0000453	0.180×10^{-7}
	0.001	0.0000453999	0.188×10^{-10}
-100	1.0	-0.485004	0.485004
	0.1	0.000030	-0.304×10^{-4}
	0.01	0.192396×10^{-44}	0.354×10^{-43}
	0.001	0.370538×10^{-43}	0.147×10^{-45}

and

$$z_{n+1} = z_n + \frac{h\lambda}{12}(5z_n + 8z_{n+1} - \hat{z}_{n+2}). \tag{5.2}$$

We eliminate $\hat{z}_{n+2}$ from (5.1) and (5.2) and solve the resulting equation for z_{n+1} in the form

$$z_{n+1} = R(\mu)z_n, \quad n = 0, 1, \ldots, \tag{5.3}$$

where

$$\mu = h\lambda, \; R(\mu) = \frac{(1 - \frac{1}{6}\mu^2)}{(1 - \mu + \frac{1}{3}\mu^2)}. \tag{5.4}$$

Consequently, our method is absolutely stable in a region in the complex $h\lambda$ plane in which (see [1,3])

$$|R(\mu)| < 1. \tag{5.5}$$

An interval (α, β) of the real line is said to be an interval of absolute stability for a numerical method if it is absolutely stable for all real values of $h\lambda \in (\alpha, \beta)$, $\lambda < 0$. We shall first determine the interval of absolute stability of the present method by examining the properties of the rational function $R(\mu)$. It is easy to see that $0 < R(\mu) < 1$ and monotonic decreasing if $\mu \in (-\sqrt{6}, 0)$ and $-\frac{1}{2} < R(\mu) < 0$ for $\mu \in (-\infty, -\sqrt{6})$. Thus, $z_n \to 0$ monotonically provided $\mu \in (-\sqrt{6}, 0)$, and z_n oscillates and tends to zero for $\mu \in (-\infty, -\sqrt{6})$ as $n \to \infty$. In either case, the interval of absolute stability is $(-\infty, 0)$, and the method is $A(0)$-stable ([3], p. 237).

Example 2. $y' = \lambda y$; $y(0) = 1$, $0 \leq x \leq 1$ has the exact solution $y(x) = e^{\lambda x}$.

For $\lambda = -100$ and $h = 0.1$, the sequence z_n oscillates (see Tables 2 and 3). This is because $h\lambda \in (-\infty, -\sqrt{6})$.

Table 3.

x	z_n	Error
0.1	-0.353383458	0.35
0.2	0.124879868	-0.12
0.3	-0.044130480	0.44×10^{-1}
0.4	0.015594981	-0.16×10^{-1}
0.5	-0.005511008	0.55×10^{-2}
0.6	0.001947499	-0.19×10^{-2}
0.7	-0.000688214	0.69×10^{-3}
0.8	0.000243203	-0.24×10^{-3}
0.9	-0.0000859441	0.86×10^{-4}
1.0	0.0000303712	-0.30×10^{-4}

6. A-STABILITY

Here we shall show that our extended trapezoidal rule is A-stable. This will be achieved by proving that the rational function $R(\mu)$ given by (5.4) satisfies $|R(\mu)| < 1$ for $\mathrm{Re}\,(\mu) < 0$.

Let $\mu = \rho e^{i\theta}$; then we have

$$
\begin{aligned}
|R(\mu)|^2 &= R(\mu)R(\bar{\mu}) \\
&= \frac{(1 - \frac{1}{6}\rho^2 e^{2i\theta})(1 - \frac{1}{6}\rho^2 e^{-2i\theta})}{(1 - \rho e^{i\theta} + \frac{1}{3}\rho^2 e^{2i\theta})(1 - \rho e^{-i\theta} + \frac{1}{3}\rho^2 e^{-2i\theta})} \\
&= \frac{1 - \frac{1}{3}\rho^2 \cos 2\theta + \frac{1}{36}\rho^4}{1 - 2\rho\cos\theta + \rho^2(1 + \frac{2}{3}\cos 2\theta) - \frac{2}{3}\rho^3\cos\theta + \frac{1}{9}\rho^4} = \frac{N}{D}\ \text{(say)}.
\end{aligned}
$$

Obviously, $N, D > 0$ and

$$
\begin{aligned}
D - N &= -2\rho\cos\theta + \rho^2(1 + \cos 2\theta) - \tfrac{2}{3}\rho^3\cos\theta + \tfrac{1}{12}\rho^4 \\
&= -2\,\mathrm{Re}\,(\mu) + 2[\mathrm{Re}\,(\mu)]^2 - \tfrac{2}{3}\rho^2\,\mathrm{Re}\,(\mu) + \tfrac{1}{12}\rho^4
\end{aligned}
$$

provides that $D - N > 0$ if $\mathrm{Re}\,(\mu) < 0$. Equivalently, we find that

$$|R(\mu)| = \sqrt{\frac{N}{D}} < 1$$

provided $\mathrm{Re}\,(\mu) < 0$.

Further, since $(D - N)/D \to \frac{3}{4}$ and $N/D = (1 - (D - N)/D) \to \frac{1}{4}$ as $\rho \to \infty$ and $\rho\cos\theta < 0$ it follows that

$$|R(\mu)| \to \tfrac{1}{2} \quad \text{as } \rho \to \infty \text{ and } \rho\cos\theta < 0. \tag{6.1}$$

From (6.1) we conclude that the present method is not L-stable ([3], pg. 236). We summarize the preceding results in the following.

THEOREM 2

The method defined by (2.3) and (2.5) for the numerical integration of the initial-value problem (1.1) is A-stable, but not L-stable.

7. APPLICATION TO THE SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

An application of our extended trapezoidal rule to the system of differential equations

$$\frac{dW}{dx} = AW, \quad W(a) = \alpha \tag{7.1}$$

where $W = [y^1, y^2, \ldots, y^n]^T$ and $A = (a_{ij})$ (an $n \times n$ constant matrix) gives

$$W_{n+1} = M(hA)W_n, \quad n = 0, 1, 2, \ldots, \tag{7.2}$$

where the transition matrix $M(hA)$ is given by

$$M(hA) = (I - hA + \tfrac{1}{3}h^2A^2)^{-1}(I - \tfrac{1}{6}h^2A^2). \tag{7.3}$$

If the present A-stable method is applied to the system (7.1) and all the eigenvalues λ_i, $i = 1(1)n$ of the matrix A satisfy $\mathrm{Re}\,(\lambda_i) < 0$, then no stability restriction on the step size h is required. However, for the monotonic convergence as discussed in section 5, it is necessary

Table 4.

h	Max. error		Min. error	
	z_A^1	z_n^2	z_n^1	z_n^2
$\frac{1}{4}$	0.393	-0.589	0.47×10^{-6}	0.71×10^{-6}
$\frac{1}{8}$	0.148	-0.222	0.63×10^{-7}	0.94×10^{-7}
$\frac{1}{10}$	0.095	-0.143	0.33×10^{-7}	0.49×10^{-7}
$\frac{1}{16}$	0.031	-0.046	0.81×10^{-8}	0.12×10^{-7}
$\frac{1}{100}$	0.740×10^{-4}	-0.111×10^{-3}	0.34×10^{-10}	0.51×10^{-10}

that

$$h[\min_{1 \le i \le n} \operatorname{Re}(\lambda_i)] \in (-\sqrt{6}, 0). \tag{7.4}$$

On the other hand, if $h \operatorname{Re}(\lambda_i) \in (-\infty, -\sqrt{6})$ then the convergence is oscillatory.

Finally, if we apply our method to the special class of problems

$$\frac{\mathrm{d}W}{\mathrm{d}x} = AW + V(x), \quad W(a) = \alpha \tag{7.5}$$

then we easily obtain

$$W_{n+1} = (I - hA + \tfrac{1}{3}h^2A^2)^{-1}[(I - \tfrac{1}{6}h^2A^2)W_n - hAR + S],$$

where the vectors R and S are given by

$$R = \frac{h}{12}(2V_n + 4V_{n+1}), \quad S = \frac{h}{12}(5V_n + 8V_{n+1} - V_{n+2}), \quad V_i \equiv V(x_i).$$

Example 3. Consider the system

$$\frac{\mathrm{d}y^1}{\mathrm{d}x} = -10y^1 + 6y^2,$$

$$\frac{\mathrm{d}y^2}{\mathrm{d}x} = 13.5y^1 - 10y^2,$$

$$y^1(0) = \tfrac{4}{3}\mathrm{e}, \quad y^2(0) = 0,$$

for which $y^1(x) = \frac{2}{3}\mathrm{e}(\mathrm{e}^{-x} + \mathrm{e}^{-19x})$, $y^2(x) = \mathrm{e}(\mathrm{e}^{-x} - \mathrm{e}^{-19x})$ is the exact solution (see [6], eqns (57), (58)). The numerical results over the range of integration[0, 10] are presented in Table 4.

Table 5.

h	Max. error		Min. error	
	z_n^1	z_n^2	z_n^1	z_n^2
$\frac{1}{4}$	0.308	-0.307	0.421×10^{-4}	0.422×10^{-4}
$\frac{1}{8}$	0.165	-0.165	0.559×10^{-5}	0.559×10^{-5}
$\frac{1}{10}$	0.121	-0.121	0.289×10^{-5}	0.289×10^{-5}
$\frac{1}{16}$	0.051	-0.051	0.721×10^{-6}	0.721×10^{-6}
$\frac{1}{50}$	0.230×10^{-2}	-0.230×10^{-2}	0.241×10^{-7}	0.241×10^{-7}

Example 4. Consider the system

$$\frac{dy^1}{dx} = -15.5y^1 + 14.5y^2 - 13.5,$$

$$\frac{dy^2}{dx} = 14.5y^1 - 15.5y^2 + 16.5,$$

$$y^1(0) = 3, \quad y^2(0) = 2,$$

for which $y^1(x) = e^{-x} + e^{-30x} + 1$, $y^2(x) = e^{-x} - e^{-30x} + 2$ is the exact solution. The numerical results over the range of integration[0,4] are presented in Table 5.

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